GLOBAL EXPONENTIAL SAMPLED-DATA OBSERVERS FOR NONLINEAR SYSTEMS WITH DELAYED MEASUREMENTS

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OUTLINE

- **Motivation**

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- Motivation
- A General Result

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- A General Result
- Globally Lipschitz Systems

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- **Systems with a compact GAS set**

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- Motivation
- A General Result
- Globally Lipschitz Systems
- Systems with a compact GAS set
- Conclusions

Motivation

Very frequently we meet a process with:

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- Nonlinear characteristics,

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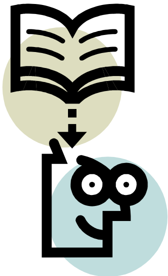
In fact, we rarely meet a system without one of the above “annoying features” !!

Motivation



QUESTION

How can we design a global exponential observer for such a system?

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Very few answers!

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- Ahmed-Ali, T., E. Cherrier, and F. Lamnabhi-Lagarrigue, "Cascade high gain predictors for nonlinear systems with delayed output ", *IEEE Transactions on Automatic Control*, 57(1), 2012, 221-226 → No sampling

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For certain classes of nonlinear systems

Motivation

A general result is needed!

A General Result

Nonlinear forward complete systems of the form:

$$\dot{x} = f(x,u), x \in \Re^n, u \in U \tag{1}$$

where $U \subseteq \Re^m$ is a non-empty set, $f : \Re^n \times U \to \Re^n$ is a smooth vector field.

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The output is given by

$$y = h(x) \tag{2}$$

where $h : \Re^n \to \Re^k$ is a smooth mapping.

A General Result

Let's put together:

$$\begin{array}{l} y(\tau_i) \approx h(x(\tau_i - r)) \\ \textit{Sampled} \\ \textit{and} \\ \textit{delayed} \\ \textit{measurements} \end{array} \rightarrow \boxed{\begin{array}{l} \textit{Robust} \\ \textit{Sampled} - \textit{Data} \\ \textit{Observer} \end{array}} \rightarrow z(t) \approx x(t-r) \rightarrow \boxed{\begin{array}{l} \text{A} \\ \text{Robust} \\ \text{Predictor} \end{array}} \rightarrow x(t)$$

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Let's put together:

$$y(\tau_i) \approx h(x(\tau_i - r))$$

Sampled

and $\rightarrow$ *Robust Sampled – Data Observer* $\rightarrow z(t) \approx x(t-r) \rightarrow$ A Robust Predictor $\rightarrow x(t)$

delayed

measurements

/ $\Downarrow$ \

A conventional observer

+ an intersample predictor

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where $F : \Re^l \times \Re^k \times U \to \Re^l$ *and* $\Psi : \Re^l \to \Re^n$ *are smooth vector fields, i.e., there exist a non-decreasing function* $M : \Re_+ \to \Re_+$ *and constants* $\sigma, \gamma > 0$ *such that for every* $(x_0, z_0, u, v) \in \Re^n \times \Re^l \times L^\infty(\Re_+; U) \times L^\infty_{loc}\left(\Re_+; \Re^k\right)$ *the solution* $(x(t), z(t))$ *of (1), (2) and (3) with initial condition* $(x(0), z(0)) = (x_0, z_0)$ *corresponding to inputs* $(u, v) \in L^\infty(\Re_+; U) \times L^\infty_{loc}\left(\Re_+; \Re^k\right)$ *exists for all* $t \geq 0$ *and satisfies the following estimate for all* $t \geq 0$*:*

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$$|\hat{x}(t) - x(t)| \leq e^{-\sigma t} M(|x_0| + |z_0| + \|u\|) + \gamma \sup_{0 \leq s \leq t} \left(e^{-\sigma(t-s)} |v(s)| \right) \tag{4}$$

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$$\begin{aligned}\dot{z}(t) &= F(z(t), w(t), u(t-r)) \\ \dot{w}(t) &= L_f h(\Psi(z(t)), u(t-r))\end{aligned} \qquad (5)$$

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is forward complete for inputs $(u,w) \in L^{\infty}([-r,+\infty);U) \times L^{\infty}_{loc}([-r,+\infty);\Re^k)$.

A General Result

Assumption (H2): *There exists a non-empty subset of the functions* $z:\Re_+ \to \Re^n$ *which are absolutely continuous on every bounded interval of* $\Re_+$ *denoted by* $A(\Re_+;\Re^n)$ *such that system (1) admits a robust global exponential* $r-$*predictor for (1) with input* $z \in A(\Re_+;\Re^n)$*, i.e.,*

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$$\begin{aligned}&\dot{\xi}(t)=F_{p}\left(\xi_{t},u_{t},z(t),\dot{z}(t)\right)\\&\tilde{x}(t)=G(\xi_{t},u_{t},z(t))\\&\xi(t)\in\Re^{q},\tilde{x}(t)\in\Re^{n},u(t)\in U\subseteq\Re^{m},z(t)\in\Re^{n}\end{aligned}\qquad(7)$$

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$$\begin{array}{l} \dot{\xi}(t) = F_p\big(\xi_t, u_t, z(t), \dot{z}(t)\big) \\ \tilde{x}(t) = G(\xi_t, u_t, z(t)) \\ \xi(t) \in \Re^q, \tilde{x}(t) \in \Re^n, u(t) \in U \subseteq \Re^m, z(t) \in \Re^n \end{array} \qquad (7)$$

where $(\xi_t)(\theta) = \xi(t+\theta)$, $(u_t)(\theta) = u(t+\theta)$, *for* $\theta \in [-r,0]$, $G: C^0([-r,0];\Re^q) \times L^\infty([-r,0];U) \times \Re^n \to \Re^n$, $F_p: C^0([-r,0];\Re^q) \times L^\infty([-r,0];U) \times \Re^n \times \Re^n \to \Re^q$

A General Result

$\forall (x_0, \xi_0, u, z) \in C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^q) \times L^\infty([-r,+\infty);U) \times A(\Re_+;\Re^n)$ *the solution* $(x(t), \xi(t)) \in \Re^n \times \Re^k$ *of (1) and (7) with initial condition* $\xi(\theta) = (\xi_0)(\theta)$, $x(\theta) = (x_0)(\theta)$, $\theta \in [-r,0]$, *corresponding to inputs* $(u,z) \in L^\infty([-r,+\infty);U) \times A(\Re_+;\Re^n)$ *is unique, defined for all* $t \geq 0$ *and satisfies*

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$$\begin{aligned} |\tilde{x}(t) - x(t)| \leq e^{-\sigma t} a\left(\|x_0\| + \|\xi_0\| + \|u\| + |z(0)| + \sup_{0 \leq s \leq r} |\dot{z}(s)| \right) \\ + P \sup_{0 \leq s \leq t} \left(e^{-\sigma(t-s)} |z(s) - x(s-r)| \right) \end{aligned}, \quad \forall t \geq 0 \qquad (8)$$

A General Result

Moreover, for every $(z_0, u, w) \in \Re^l \times L^\infty([-r,+\infty);U) \times L^\infty_{loc}([-r,+\infty);\Re^k)$*, the output signal* $\hat{x}(t) = \Psi(z(t))$ *produced by the unique solution of (6) with initial condition* $z(0) = z_0$ *and corresponding to inputs* $(u,w) \in L^\infty([-r,+\infty);U) \times L^\infty_{loc}([-r,+\infty);\Re^k)$ *is a function of class* $A(\Re_+;\Re^n)$.

A General Result

Assumption (H3): *There exist a constant* $C > 0$, *a continuous function* $T : \Re^n \times \Re^l \to \Re_+$ *and a non-decreasing function* $N : \Re_+ \to \Re_+$ *such that for every* $(x_0, z_0, u, v) \in \Re^n \times \Re^l \times L^\infty(\Re_+; U) \times L^\infty_{loc}\left(\Re_+; \Re^k\right)$ *the solution* $(x(t), z(t))$ *of (1), (2) and (3) with initial condition* $(x(0), z(0)) = (x_0, z_0)$ *corresponding to inputs* $(u, v) \in L^\infty(\Re_+; U) \times L^\infty_{loc}\left(\Re_+; \Re^k\right)$ *satisfies the following estimate for all* $t \geq T(x_0, z_0)$*:*

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$$\begin{aligned} &\left| L_f h(\hat{x}(t), u(t)) - L_f h(x(t), u(t)) \right| \\ &\leq e^{-\sigma t} N\left(|x_0| + |z_0| + \|u\| \right) + C \sup_{0 \leq s \leq t} \left(e^{-\sigma(t-s)} |v(s)| \right) \end{aligned} \tag{9}$$

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Theorem: *Consider system (1) under hypotheses (H1-3). Let* $0 < b \leq B$ *be (arbitrary) constants satisfying:*

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Then there exists a non-decreasing function $Q : \Re_+ \to \Re_+$ *such that for every partition* $\pi = \{\tau_i\}_{i=0}^{\infty}$ *of* $\Re_+$ *with* $\sup_{i \ge 0}(\tau_{i+1} - \tau_i) \le B$ *and* $\inf_{i \ge 0}(\tau_{i+1} - \tau_i) \ge b$, *for every* $(z_0, w_0, u, v) \in \Re^l \times \Re^k \times L^{\infty}([-r,+\infty); U) \times L^{\infty}_{loc}(\Re_+; \Re^k)$, $(x_0, \xi_0) \in C^0([-r,0]; \Re^n) \times C^0([-r,0]; \Re^q)$ *the unique solution of the system (1) with*

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(the conventional robust observer)

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(the intersample predictor)

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$$w(\tau_{i+1}) = h(x(\tau_{i+1} - r)) + v(\tau_{i+1})$$

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$$\begin{aligned}\dot{\xi}(t) &= F_p\left(\xi_t, u_t, \hat{x}(t), \frac{d\hat{x}}{dt}(t)\right) \\ \tilde{x}(t) &= G(\xi_t, u_t, \hat{x}(t))\end{aligned}$$

(the robust predictor)

A General Result

with initial condition $\xi(\theta)=\left(\xi_0\right)(\theta)$, $x(\theta)=\left(x_0\right)(\theta)$, $\theta\in[-r,0]$, $(z(0),w(0))=(z_0,w_0)$ *corresponding to inputs* $(u,v)\in L^{\infty}\left([-r,+\infty);U\right)\times L^{\infty}_{loc}\left(\Re_+;\Re^k\right)$ *is defined for all* $t\geq 0$ *and satisfies the estimate:*

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$$\begin{aligned} & |\tilde{x}(t) - x(t)| \\ & \leq e^{-\sigma t} Q\left(\|x_0\| + \|\xi_0\| + \|u\| + |z_0| + |w_0| + \sup_{0 \leq s \leq t} (|v(s)|) \right) \\ & + \frac{\gamma P \exp(\sigma B)}{1 - CB \exp(\sigma B)} \sup_{0 \leq s \leq t} \left(e^{-\sigma(t-s)} |v(s)| \right) \end{aligned}, \quad \forall t \geq 0$$

Globally Lipschitz Systems

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There exists a constant $L > 0$ *such that*

$$\boxed{\left| f(x,u) - f(z,u) \right| \le L \left| x - z \right|}, \forall x, z \in \Re^n,\ \forall u \in U$$

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There exists a symmetric, positive definite matrix $P \in \Re^{n \times n}$, *a constant* $q > 0$ *and matrices* $K \in \Re^{n \times k}$, $H \in \Re^{k \times n}$ *such that:*

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$$\boxed{(z-x)'P\big(f(z,u) - f(x,u)\big) + (z-x)'PKH(z-x) \le -q|z-x|^2},$$

$$\forall x, z \in \Re^n, \; \forall u \in U$$ **(a conventional high-gain observer)**

Globally Lipschitz Systems

Theorem: *Let* $r>0$ *be a constant. For every positive integer* $p>0$ *with* $Lr<p$, *for every* $\mu>0$, $0<b\leq B$ *with* $L|H|\frac{\sqrt{|K'PPK|}}{q}\sqrt{\frac{|P|}{R}}B<1$, *there exist a non-decreasing function* $Q:\Re_+\to\Re_+$ *and constants* $\sigma,\Gamma>0$ *such that for every partition* $\pi=\{\tau_i\}_{i=0}^{\infty}$ *of* $\Re_+$ *with* $\sup_{i\geq 0}(\tau_{i+1}-\tau_i)\leq B$ *and* $\inf_{i\geq 0}(\tau_{i+1}-\tau_i)\geq b$, *for every* $(z_0,w_0,u,v)\in\Re^n\times\Re^k\times L^{\infty}([-r,+\infty);U)\times L^{\infty}_{loc}(\Re_+;\Re^k)$, $x_0\in C^0([-r,0];\Re^n)$, $\xi_{i,0}\in C^0([-r,0];\Re^n)$ $(i=1,\ldots,p)$ *the unique solution of the system (1) with*

Globally Lipschitz Systems

$$\dot{z}(t) = f(z(t), u(t-r)) + K(Hz(t) - w(t))$$ **(the conventional observer)**

Globally Lipschitz Systems

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Globally Lipschitz Systems

and the robust cascade predictor

Globally Lipschitz Systems

and the robust cascade predictor

$$\dot{\xi}_1(t) = f(\xi_1(t), u(t-r+\delta)) - f(\xi_1(t-\delta), u(t-r)) + \dot{z}(t) - \mu\left(\xi_1(t) - z(t) - \int_{t-\delta}^{t} f\left(\xi_1(s), u(s-r+\delta)\right) ds \right)$$

$$\dot{\xi}_i(t) = f(\xi_i(t), u(t-r+i\delta)) - f(\xi_i(t-\delta), u(t-r+(i-1)\delta)) + \dot{\xi}_{i-1}(t) - \mu\left(\xi_i(t) - \xi_{i-1}(t) - \int_{t-\delta}^{t} f\left(\xi_i(s), u(s-r+i\delta)\right) ds \right)$$

$$, \; i = 2, \ldots, p$$

Globally Lipschitz Systems

with $\delta := p^{-1} r$, *initial condition* $x(\theta) = x_0(\theta)$, $\xi_i(\theta) = \xi_{i,0}(\theta)$, $\theta \in [-r, 0]$, *(*$i = 1, \ldots, p$*)*, $(z(0), w(0)) = (z_0, w_0)$ *corresponding to inputs* $(u, v) \in L^{\infty}\left([-r, +\infty); U\right) \times L^{\infty}_{loc}\left(\Re_+ ; \Re^k\right)$ *is defined for all* $t \geq 0$ *and satisfies the estimate:*

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$$

Arbitrarily large delay!

Globally Lipschitz Systems

As $r \to +\infty$ **(the measurement delay)**

then $\Gamma \to +\infty$ **(sensitivity to measurement error)**

(expected)

Systems with a Compact GAS set

Suppose that:

The set $U \subset \Re^m$ *is compact and there exist a non-empty compact set* $S \subset \Re^n$, *a continuous function* $T : \Re^n \to \Re_+$ *and a smooth positive function* $\psi : \Re^n \to (0,+\infty)$ *such that for every* $u \in L^\infty(\Re_+;U)$ *and for every initial condition* $x(0) \in \Re^n$ *the solution of (1) satisfies:*

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$$\boxed{|x(t)| \leq \psi(x(0))}, \ \forall t \geq 0$$

Systems with a Compact GAS set

Suppose also that $h(\Re^n) = \Re$ and that:

We have robust global exponential observer such that for every $(u,w) \in L^\infty([-r,+\infty);U) \times L^\infty_{loc}([-r,+\infty);\Re^k)$ *and for every initial condition* $z(0) \in \Re^n$ *it holds that:*

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(a systematic procedure for designing such observers can be found in Karafyllis, I. and C. Kravaris, “Global Exponential Observers for Two Classes of Nonlinear Systems”, *Systems and Control Letters*, 61(7), 2012, 797-806.)

Systems with a Compact GAS set

Notice that that there exists a constant $G \geq 0$ such that for every $(x_0, z_0, u, v) \in \Re^n \times \Re^l \times L^{\infty}(\Re_+; U) \times L^{\infty}_{loc}\left(\Re_+; \Re^k\right)$ the solution $(x(t), z(t))$ of (1), (2) and (3) with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to inputs $(u, v) \in L^{\infty}(\Re_+; U) \times L^{\infty}_{loc}\left(\Re_+; \Re^k\right)$ satisfies the following estimate for all $t \geq \max(T(x_0), T(z_0))$:

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$$\left| L_f h(\hat{x}(t), u(t)) - L_f h(x(t), u(t)) \right| \leq G \left| \hat{x}(t) - x(t) \right|$$

Systems with a Compact GAS set

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Systems with a Compact GAS set

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$$p(s) := \max\left\{ \left| f(\xi,u) \right| : u \in U , \left| \xi \right| \le K \max\left\{ \psi(z) : \left| z \right| \le s \right\} \right\}$$

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Let $G_1, G_2 > 0$ *be constants satisfying*

$$\left| f\left(q\left(\frac{|\xi|}{\psi(z)}\right)\xi, u\right) - f\left(q\left(\frac{|x|}{\psi(y)}\right)x, u\right)\right| \le G_1|\xi - x| + G_2|z - y|$$
$$\forall u \in U, y, x, z \in S, \xi \in \tilde{S}$$

Systems with a Compact GAS set

Theorem: *If* $G_1 r < 1$*, then for every* $\mu > 0$*,* $0 < b \le B$ *with* $G\gamma B < 1$*, where* γ *is the gain of the measurement error for the conventional observer (3), there exist a non-decreasing function* $Q : \Re_+ \to \Re_+$ *and constants* $\sigma, \Gamma > 0$ *such that for every partition* $\pi = \{\tau_i\}_{i=0}^{\infty}$ *of* $\Re_+$ *with* $\sup_{i \ge 0}(\tau_{i+1} - \tau_i) \le B$ *and* $\inf_{i \ge 0}(\tau_{i+1} - \tau_i) \ge b$*, for every* $(z_0, w_0, u, v) \in \Re^n \times \Re^k \times L^\infty([-r,+\infty);U) \times L^\infty_{loc}(\Re_+;\Re^k)$*,* $(x_0, \xi_0) \in C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^n)$ *the solution of (1) with*

Systems with a Compact GAS set

$$\begin{aligned}\dot{z}(t) &= F(z(t), w(t), u(t-r)) \\ \hat{x}(t) &= \Psi(z(t))\end{aligned}$$

(the conventional observer)

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$$\dot{z}(t) = F(z(t), w(t), u(t-r))$$
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$$\dot{w}(t) = L_f h(\hat{x}(t), u(t-r)), \ t \in [\tau_i, \tau_{i+1})$$

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$$\begin{aligned}\dot{\xi}(t) = \dot{z}(t) + \frac{d}{dt}\int_{t-\delta}^{t} f\left(q\left(\frac{|\xi(s)|}{\psi(z(t))} \right)\xi(s), u(s-r+\delta) \right) ds \\ - \mu\left(\xi(t) - z(t) - \int_{t-\delta}^{t} f\left(q\left(\frac{|\xi(s)|}{\psi(z(t))} \right)\xi(s), u(s-r+\delta) \right) ds \right)\end{aligned}$$ **(the predictor)**

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with $\delta = r$, *initial condition* $x(\theta) = x_0(\theta)$, $\xi(\theta) = \xi_0(\theta)$, $\theta \in [-r,0]$, $(z(0), w(0)) = (z_0, w_0)$ *corresponding to inputs* $(u,v) \in L^{\infty}([-r,+\infty);U) \times L^{\infty}_{loc}(\Re_+;\Re^k)$ *is defined for all* $t \geq 0$ *and satisfies the estimate:*

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Not arbitrarily large delay

Conclusions

Novel results for a practical and

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Possible use for predictor-based control of systems with delayed input

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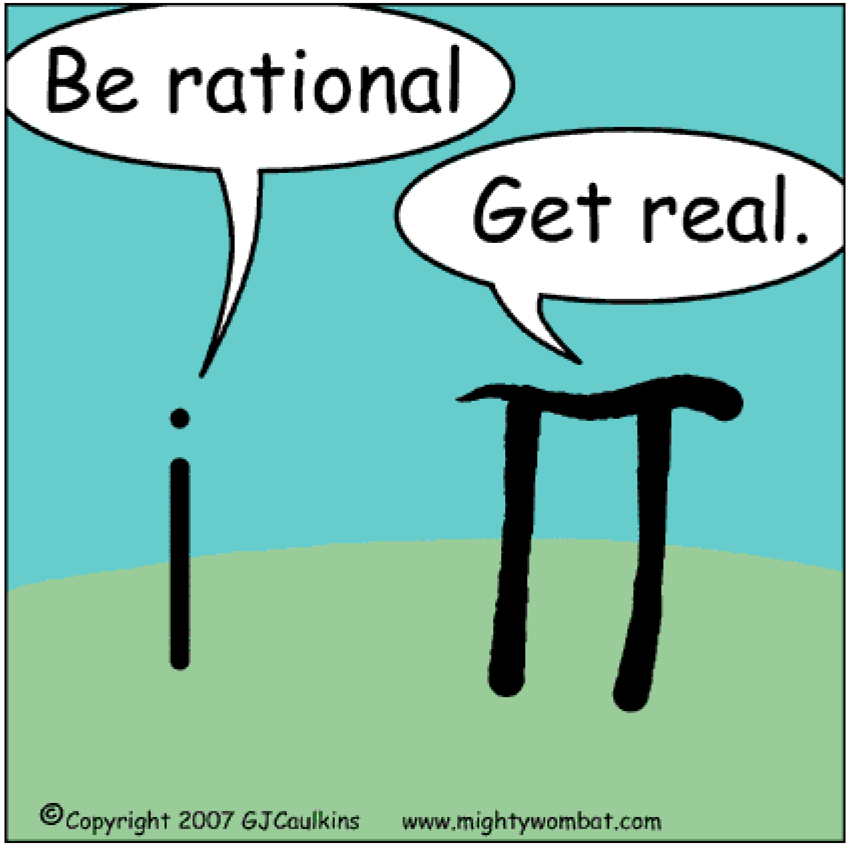
Possible extensions to other classes of nonlinear systems


Be rational
Get real.
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THANK YOU!