Advances in the stability analysis of interconnected nonlinear systems by ISS-related concepts

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Overview

Context

Essential questions in many control applications:

- Does the system eventually behave as I want it to?

Convergence

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- Does the system react moderately to small variations of its initial conditions?

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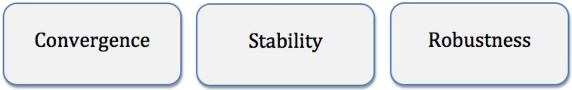

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- Does the systems interact properly with other systems?
- If yes, to what extent?

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Robustness ⇔ Convergence ⇒ Stability

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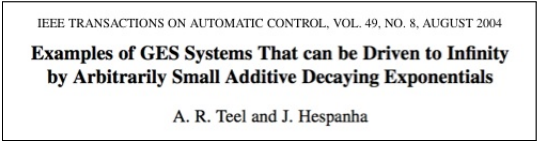
IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 8, AUGUST 2004

Examples of GES Systems That can be Driven to Infinity by Arbitrarily Small Additive Decaying Exponentials

A. R. Teel and J. Hespanha

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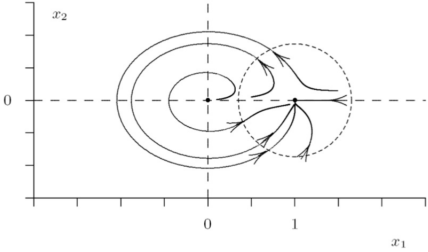

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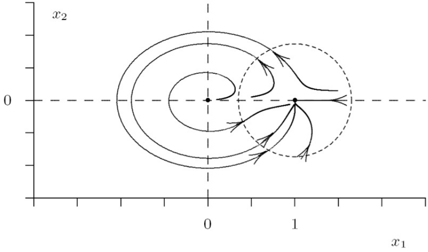
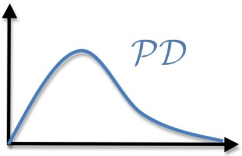


- A key tool: Input-to-State Stability (ISS)

Comparison functions



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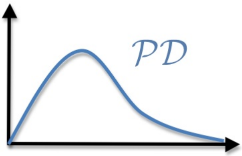

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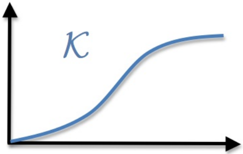

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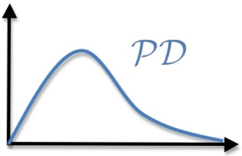

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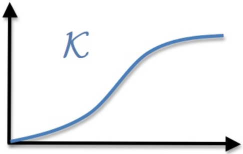

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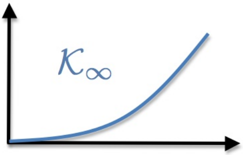

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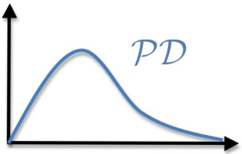

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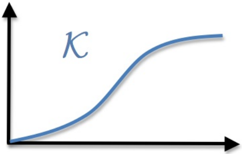

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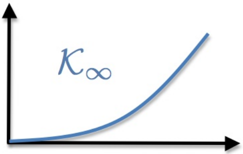

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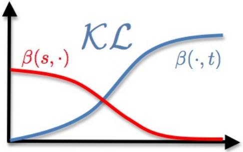

$$\begin{cases} \beta(\cdot,t) \in \mathscr{K}, \forall t \geq 0 \\ \beta(s,\cdot) \text{ nonincreasing}, \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s,t) = 0, \forall s \geq 0 \end{cases}$$

ISS and iISS notions

Definition: Input-to-State Stability (ISS), [Sontag, 1989]

The system $\dot{x} = f(x,u)$ is ISS if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}^m$,

$$|x(t;x_0,u)| \leq \beta(|x_0|,t) + \gamma(\|u\|), \quad \forall t \geq 0.$$

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→ Measures the impact of input energy

Strengths and weaknesses

ISS and iISS: Central tools in analysis and control:

- Theoretical contributions to: output feedback, optimal control, hybrid systems, predictive control, chaotic systems...
- Applications in: robotics, production lines, transportation, bio-chemical networks, control under communication constraints, neuroscience, ...

ISS	iISS
$\dot{x} = f(x,0)$ is GAS	$\dot{x} = f(x,0)$ is GAS
Bounded input $\Rightarrow$ Bounded state	Bounded energy input $\Rightarrow$ Bounded, converging state
Converging input $\Rightarrow$ Converging state	Small inputs may yield unbounded state
Cascade: ISS + ISS $\Rightarrow$ ISS	Converging input $\not\Rightarrow$ Converging state
	Cascade: iISS + iISS $\not\Rightarrow$ iISS

In practice, some systems do exhibit robustness for inputs under a certain threshold, but diverge for larger ones.

Strong iISS: halfway between ISS and iISS.

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The system $\dot{x} = f(x,u)$ is Strongly iISS if it is:

- iISS
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i.e., if there exist $\beta \in \mathcal{KL}$, $\mu_1, \mu_2, \gamma \in \mathcal{K}_\infty$ and input threshold $R > 0$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}^m$,

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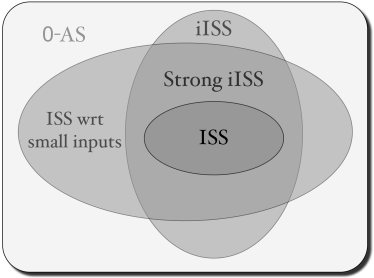

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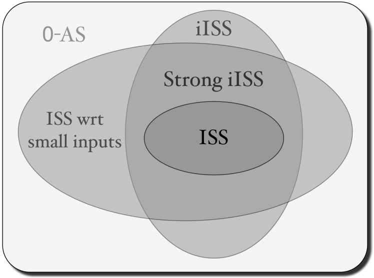

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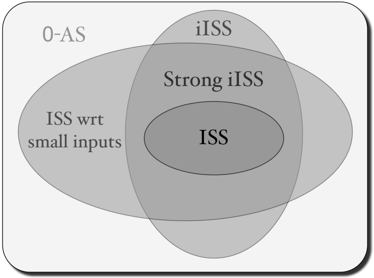

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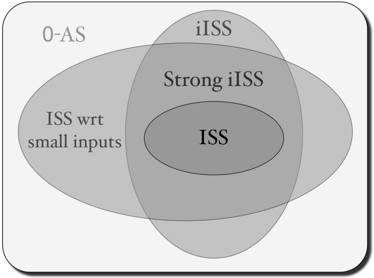

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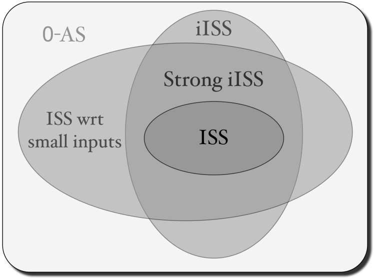

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Lyapunov characterization of ISS and iISS

- Part of the success of ISS and iISS is due to their Lyapunov characterization
- Lyapunov function candidate (LFC):
  - $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable
  - $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$
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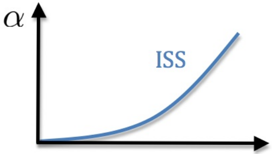
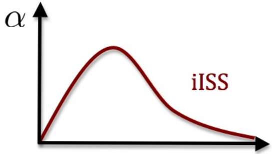
ISS and iISS characterization, [Sontag & Wang 1995, Angeli et al. 2000]

The system $\dot{x} = f(x,u)$ is ISS (resp. iISS) if and only if there exist a LFC V, $\gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{K}_\infty$ (resp. $\alpha \in \mathcal{PD}$) such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$





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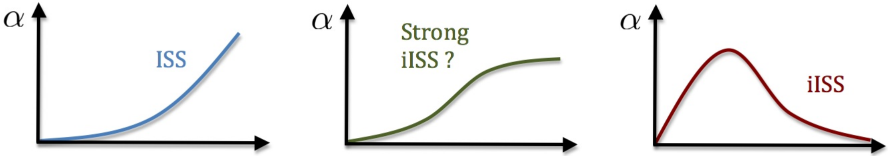
  - $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$
  - $V(x) \to \infty$ whenever $|x| \to \infty$.

ISS and iISS characterization, [Sontag & Wang 1995, Angeli et al. 2000]

The system $\dot{x} = f(x,u)$ is ISS (resp. iISS) if and only if there exist a LFC V, $\gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{K}_\infty$ (resp. $\alpha \in \mathcal{PD}$) such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$



Lyapunov conditions for Strong iISS

Theorem: $\mathscr{K}$ dissipation rate $\Rightarrow$ Strong iISS

If there exists a CLF $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying,for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

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where $\alpha \in \mathscr{K}$ and $\gamma \in \mathscr{K}_\infty$, then the system $\dot{x} = f(x,u)$ is Strongly iISS with input threshold $R = \gamma^{-1} \circ \alpha(\infty)$.

However, the converse does not hold:

Counter-example: Strong iISS $\not\Rightarrow$ $\mathscr{K}$ dissipation rate

The scalar system

$$\dot{x} = -\frac{x}{1+x^2} + r(|u|-1),$$

where r is the unit ramp, is Strongly iISS. However, for all $\alpha \in \mathscr{K}$ and $\gamma \in \mathscr{PD}$ no differentiable function $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies

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Theorem: Lyapunov condition for Strong iISS

Assume that there exist a LFC $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha, \gamma \in \mathscr{K}$, $R_x, R_u \geq 0$ and three continuous nondecreasing functions $\eta, \nu_1, \nu_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$|u| \leq R_u \quad \Rightarrow \quad \frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(|x|) + \gamma(|u|) \tag{1}$$

$$|x| \geq R_x \quad \Rightarrow \quad \frac{\partial V}{\partial x}(x) f(x,u) \leq \nu_1(|u|) V(x) + \nu_2(|u|). \tag{2}$$

Then the system $\dot{x} = f(x,u)$ is Strongly iISS.

- Condition (1): ISS w.r.t. small inputs
- Condition (2): forward completeness [Angeli & Sontag 1999]
- Note: ISS w.r.t. small inputs + forward completeness is not enough for Strong iISS [Angeli et al. 2000]

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Corollary

Assume that there exist a LFC $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\lambda, \gamma \in \mathcal{K}_\infty$, and $q > 0$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x}(x)f(x,u) \leq -(q - \lambda(|u|))V(x) + \gamma(|u|).$$

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Extension of [Theorem 2, Sontag 1998] where iISS is established under the same condition.

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Strong iISS for particular classes of systems

Theorem: Strong iISS for perturbed LTI systems

Let φ be a locally Lipschitz function satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$|\varphi(x,u)| \leq (c_1 + c_2 |x|) g(|u|)$$

with $c_1, c_2 \geq 0$ and $g \in \mathscr{PD}$. Then the system

$$\dot{x} = Ax + \varphi(x,u),$$

is Strongly iISS if and only if $A \in \mathbb{R}^{n \times n}$ is Hurwitz.

- If $c_2 = 0$, then ISS
- Special case: bilinear systems

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i A_i \right) x + Bu,$$

for which iISS was established in [Sontag 1998, Theorem 5].

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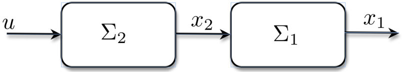
Cascades of Strong iISS systems



$$\begin{aligned} \Sigma_1 : \quad \dot{x}_1 &= f_1(x_1, x_2) && \text{(3a)} \\ \Sigma_2 : \quad \dot{x}_2 &= f_2(x_2, u)\,. && \text{(3b)} \end{aligned}$$

- ISS is naturally preserved in cascade [Sontag & Teel 1995]
- iISS is not preserved by cascade [Panteley & Loría 2001, Arcak et al. 2002, Chaillet & Angeli 2008].

Theorem: Strong iISS is preserved under cascade

If the systems $\dot{x}_1 = f_1(x_1, u_1)$ and $\dot{x}_2 = f_2(x_2, u_2)$ are Strongly iISS, then the cascade (3) is Strongly iISS.

Corollary: iISS + Strong iISS $\Rightarrow$ iISS

If $\dot{x}_1 = f_1(x_1, u_1)$ is Strongly iISS and $\dot{x}_2 = f_2(x_2, u_2)$ is iISS, then (3) is iISS.

Corollary: GAS + Strong iISS $\Rightarrow$ GAS

If $\dot{x}_1 = f_1(x_1, u_1)$ is Strongly iISS and $\dot{x}_2 = f_2(x_2)$ is GAS, then

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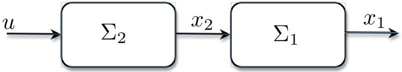

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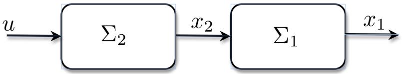

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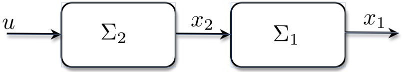

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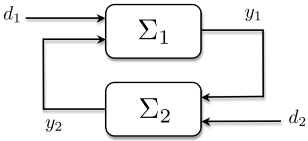
Feedback interconnection of ISS systems



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Theorem: Small gain for ISS [Jiang et al. 1996]

For each $i \in \{1,2\}$, assume that there exist a LFC $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and $\alpha_i, \gamma_i, \chi_i \in \mathcal{K}_\infty$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $(d_1, d_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$,

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If $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$, then the feedback system is ISS.

- Several variants: e.g. [Jiang et al. 1994, Teel 1996, Karafyllis & Tsinias 2004]
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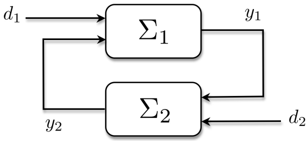

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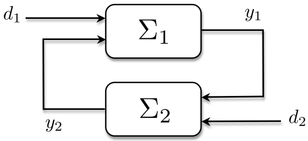

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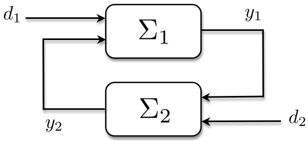

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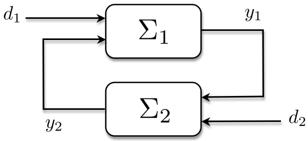

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For each $i \in \{1,2\}$, assume that there exist a LFC $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and $\alpha_i, \gamma_i, \chi_i \in \mathcal{K}_\infty$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $(d_1, d_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$,

$$V_1(x_1) \geq \max\{\chi_1(V_2(x_2)); \gamma_1(|d_1|)\} \quad \Rightarrow \quad \frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, x_2, d_1) \leq -\alpha_1(|x_1|)$$

$$V_2(x_2) \geq \max\{\chi_2(V_1(x_1)); \gamma_2(|d_2|)\} \quad \Rightarrow \quad \frac{\partial V_2}{\partial x_2}(x_2) f_2(x_1, x_2, d_2) \leq -\alpha_2(|x_2|).$$

If $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$, then the feedback system is ISS.

- Several variants: e.g. [Jiang et al. 1994, Teel 1996, Karafyllis & Tsinias 2004]
- Extension to multiple feedback loops: [Dashkovskiy et al. 2010, Liu et al. 2011].

Feedback interconnection of Strong iISS systems

Small gain for Strong iISS [Ito & Jiang 2009]

For each $i \in \{1,2\}$, assume that there exist a LFC V_i and $\alpha_i, \gamma_i, \sigma_i \in \mathcal{K}$ such that, for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$,

$$\frac{\partial V_1}{\partial x_1}(x_1) \leq -\alpha_1(V_1(x_1)) + \gamma_1(V_2(x_2)) + \sigma_1(|d_1|)$$
$$\frac{\partial V_2}{\partial x_2}(x_2) \leq -\alpha_2(V_2(x_2)) + \gamma_2(V_1(x_1)) + \sigma_2(|d_2|).$$

Assume further that there exist $\omega_1, \omega_2 \in \mathcal{K}_\infty$ such that, for all $s \in \mathbb{R}_{\geq 0}$,

$$\alpha_1^{-1} \circ (\mathrm{id} + \omega_1) \circ \gamma_1 \circ \alpha_2^{-1} \circ (\mathrm{id} + \omega_2) \circ \gamma_2(s) \leq s\,.$$

Then the two following facts hold:

- If $d \equiv 0$, then the feedback system is GAS;
- If, for each $i \in \{1,2\}$,

$$\lim_{s\to\infty} \alpha_i(s) = \infty \quad \text{or} \quad \lim_{s\to\infty} \gamma_{3-i}(s) < \infty,$$

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Feedback interconnection of Strong iISS systems

- Simplified and tight versions for systems without inputs: [Angeli & Astolfi 2006, Praly & Astolfi 2012]
- In presence of exogenous inputs, no Lyapunov-based small-gain theorem dissipation inequalities can be expected if α_1 or α_2 is not in class $\mathscr{K}$: Strong iISS is necessary in that sense
- Extension to networks of multiple interconnected Strong iISS systems: [Ito et al. 2009, Ito et al. 2013]

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So far:

- Strong iISS: halfway between iISS and ISS
- Intuitive Lyapunov characterization (class $\mathcal{K}$ dissipation rate) is not a necessary requirement
- Alternative sufficient conditions are provided
- Strong iISS is well behaved in cascades
- Feedback interconnection:
  - Small-gain for Strongly iISS systems [Ito 2006]
  - Small-gain for large scale networks [Ito et al. 2011]

Perspective:

- Lyapunov characterization is still missing
- Envisioned applications:
  - Control with quantized measurements
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