Control Hierarchies and Tropical Algebras

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Outline

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Motivation and Aims

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- Want to address large-scale/complex control problems
- Too many degrees of freedom for monolithic controller design
- Need to impose structure to reduce degrees of freedom
- Hierarchical control architecture particularly intuitive
- Heuristically designed hierarchical control ubiquituos in industry

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- Hierarchical structures need not be “rigid”; may be embedded into consensus-type distributed systems, with top-level functionality temporarily assigned to a node

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Abstraction and Refinement

- Have been investigated in different scenarios
- Behavioural point of view allows conceptionally (and notationally) simple explanation of main ingredients

Dynamical system with input/output structure:

$$\Sigma = \left(T, U \times Y, \mathfrak{B} \subseteq (U \times Y)^T\right)$$

Abstractions and refinements:

- $\Sigma_a = (T, U \times Y, \mathfrak{B}_a)$ is an abstraction of Σ if $\mathfrak{B} \subseteq \mathfrak{B}_a$
- $\Sigma_r = (T, U \times Y, \mathfrak{B}_r)$ is a refinement of Σ if $\mathfrak{B}_r \subseteq \mathfrak{B}$

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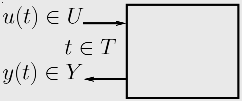

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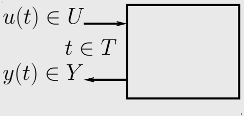

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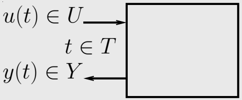

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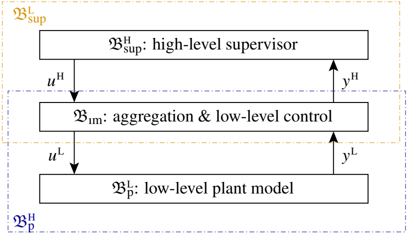
Generic Two-Level Control Structure



. . . can be extended to arbitrary number of control layers . . .

- Low-level signal space: $W_{\mathrm{L}} = U_{\mathrm{L}} \times Y_{\mathrm{L}}$.
- Low-level process model: $\mathfrak{B}^{\mathrm{L}}_{\mathrm{p}}$ . . . behaviour on W_{L}.
- Inclusion-type specification: $\mathfrak{B}^{\mathrm{L}}_{\mathrm{spec}}$ . . . defined on W_{L}.
- High-level signal space: $W_{\mathrm{H}} = U_{\mathrm{H}} \times Y_{\mathrm{H}}$.
- High-level supervisor: $\mathfrak{B}^{\mathrm{H}}_{\mathrm{sup}}$ . . . behaviour on W_{H}.
- Low-level control: $\mathfrak{B}_{\mathrm{lm}}$ . . . behaviour on $W_{\mathrm{H}} \times W_{\mathrm{L}}$.

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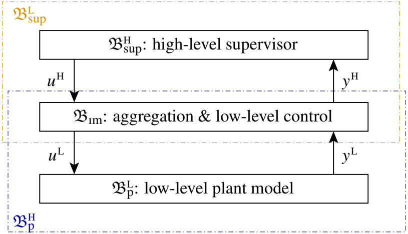

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- High-level supervisor: $\mathfrak{B}^H_{sup}$ …behaviour on W_H.
- Low-level control: $\mathfrak{B}_{IM}$ …behaviour on $W_H \times W_L$.

Design Procedure

Define high-level signal space (assumed given in this talk).

Low-level control:

- Define (inclusion-type) specs $\mathfrak{B}^{\mathrm{HL}}_{\mathrm{spec}}$ for lower control layer – *intended* relation between high-level and low-level signals.
- Design low-level control $\mathfrak{B}_{\mathsf{im}}$ enforcing specs $\mathfrak{B}^{\mathrm{HL}}_{\mathrm{spec}}$.

High-level control:

Synthesise $\mathfrak{B}^{\mathrm{H}}_{\mathrm{sup}}$ for $\mathfrak{B}^{\mathrm{H}}_{\mathrm{p}} = \mathfrak{B}^{\mathrm{H}}_{\mathsf{im}}[\mathfrak{B}^{\mathrm{L}}_{\mathrm{p}}]$. Can be done abstraction-based!

- Use high-level proj. $\mathrm{P}^{\mathrm{H}}(\mathfrak{B}^{\mathrm{HL}}_{\mathrm{spec}})$ of $\mathfrak{B}^{\mathrm{HL}}_{\mathrm{spec}}$ as abstraction of $\mathfrak{B}^{\mathrm{H}}_{\mathrm{p}}$.
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Where Can Things Go Wrong?

Low-level specification $\mathfrak{B}_{\text{spec}}^{\text{HL}}$ too demanding:

- I.e., we cannot find appropriate low-level control.
- Need to relax low-level specifications and replace $\mathfrak{B}_{\text{spec}}^{\text{HL}}$ by an abstraction $\mathfrak{B}_{\text{spec,a}}^{\text{HL}}$ such that $\mathfrak{B}_{\text{spec}}^{\text{HL}} \subseteq \mathfrak{B}_{\text{spec,a}}^{\text{HL}}$.

Illustration: robot moving in a restricted area:

$$\begin{aligned} \dot{x}_1(t) &= v(t)\cos\theta(t) \\ \dot{x}_2(t) &= v(t)\sin\theta(t) \\ \dot{\theta}(t) &= u_1(t) \\ \dot{v}(t) &= u_2(t) \end{aligned}$$

$u^{\text{L}} = (u_1, u_2)$ low-level inputs
$y^{\text{L}} = (x_1, x_2)$ low-level outputs
$u^{\text{H}} \in \{\text{go up}, \ldots\}$ high-level input
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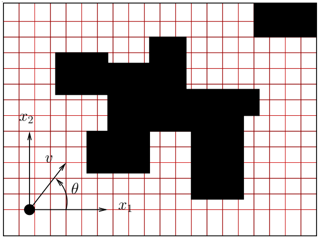

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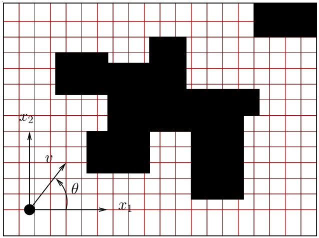

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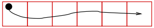
Ex.: "go right" $\rightsquigarrow$ 
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Recap:

- choice of low-level specs $\mathfrak{B}^{\text{HL}}_{\text{spec}}$ depends on engineering intuition
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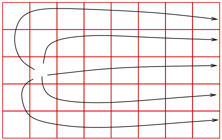

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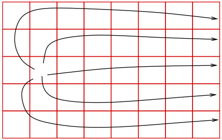

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Specific Scenario

- top layer decides on timing (not ordering!) of discrete events
- synthesis based on TEG abstraction of plant + low-level control
- TEG (Timed Event Graph) . . . specific timed Petri net

Example:

$$x_7(k) = \max\{x_4(k) + 1, x_2(k) + 6, x_2(k+1), x_8(k-1)\}$$

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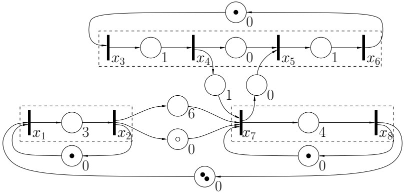

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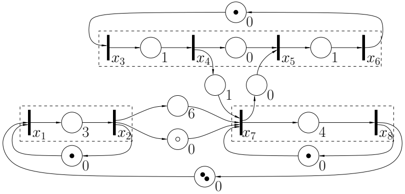

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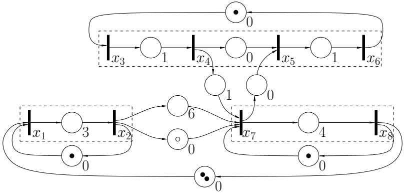

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Dioid Algebras

A dioid is an algebraic structure with two binary operations $\oplus$ ("addition") and $\otimes$ ("multiplication") defined on a set $\mathcal{D}$, such that

- $\oplus$ is associative, commutative & idempotent ($a \oplus a = a \; \forall a \in \mathcal{D}$)
- $\otimes$ is associative and is distributive w.r.t. $\oplus$
- zero element ε, unit element e
- ε is absorbing for $\otimes$, i.e., $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon \; \forall a \in \mathcal{D}$

Remarks

- a dioid is complete if it is closed for infinite sums and $\otimes$ distributes over infinite sums
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Example: The Max-Plus Algebra

Defined on $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$ resp. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$:

- addition: $a \oplus b := \max(a, b)$, zero element: $\varepsilon := -\infty$
- multiplication: $a \otimes b := a + b$, unit element: $e := 0$

Time relations for TEGs described by linear implicit difference eqns.

For our example

$$x_7(k) = 1 \otimes x_4(k) \oplus 6 \otimes x_2(k) \oplus x_2(k+1) \oplus x_8(k-1)$$

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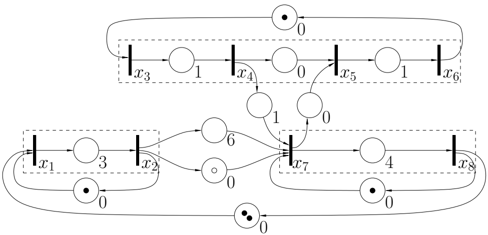

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The Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

- $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ ... a quotient dioid in the set of 2-dim. formal power series (in γ, δ), with Boolean coefficients and integer exponents
- interpretation of monomial $\gamma^k\delta^t$:
  - kth occurrence of event is at time t at the earliest
  - equivalently: at time t, event has occurred at most k times

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Properties:

- $\gamma^k\delta^t \oplus \gamma^l\delta^t = \gamma^{\min(k,l)}\delta^t$
- $\gamma^k\delta^t \oplus \gamma^k\delta^\tau = \gamma^k\delta^{\max(t,\tau)}$
- $\gamma^k\delta^t \otimes \gamma^l\delta^\tau = \gamma^{(k+l)}\delta^{(t+\tau)}$
- Zero element: $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$
- Unit element: $e = \gamma^0\delta^0$
- interpretation of partial order: inclusion in $\overline{\mathbb{Z}}^2$

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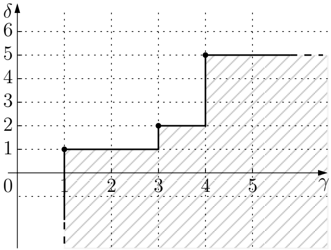
$\rightsquigarrow$ have to consider "south-east cones" (instead of points) in $\overline{\mathbb{Z}}^2$

Example: $s = \gamma^1\delta^1 \oplus \gamma^3\delta^2 \oplus \gamma^4\delta^5$



Properties:

- $\gamma^k\delta^t \oplus \gamma^l\delta^t = \gamma^{\min(k,l)}\delta^t$
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- $\gamma^k\delta^t \otimes \gamma^l\delta^\tau = \gamma^{(k+l)}\delta^{(t+\tau)}$
- Zero element: $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$
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- interpretation of partial order: inclusion in $\overline{\mathbb{Z}}^2$

The Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

- $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ ... a quotient dioid in the set of 2-dim. formal power series (in γ, δ), with Boolean coefficients and integer exponents
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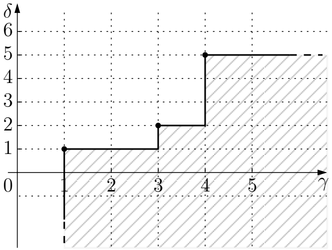

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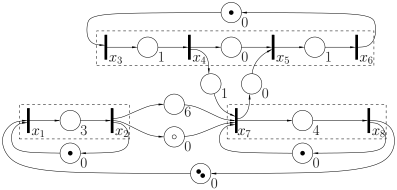
The Dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ ctd.

Time relations for TEGs become linear algebraic eqns. in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

For our example



$$x_7 = \delta^1\gamma^0 x_4 \oplus (\delta^6\gamma^0 \oplus \delta^0\gamma^{-1})x_2 \oplus \delta^0\gamma^1 x_8$$

In general, with input & output trans. (triggered resp. seen externally):

$$\begin{aligned} x &= Ax \oplus Bu \\ y &= Cx \end{aligned}$$

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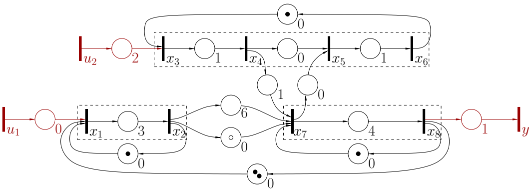

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Control in the Dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

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- state model $x = Ax \oplus Bu,\ y = Cx$
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Output feedback:

$$\begin{aligned} u &= Ky \oplus v \\ \rightsquigarrow y &= CA^*BKy \oplus CA^*Bv \\ y &= \underbrace{(CA^*BK)^*CA^*B}_{H_{cl}}\, v \end{aligned}$$

Aim: just-in-time policy

find greatest K s.t. $H_{ref} \succeq H_{cl}$, with

- H_{ref} a given reference model
- "greatest" and "$\succeq$" in the sense of natural order in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

Solution:

desired feedback K can be obtained using "residuation theory":

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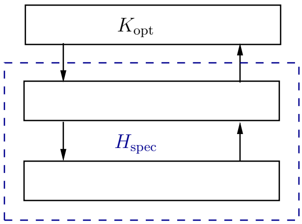
Outline

Tradeoff Between Control Layers



- K_{opt} ... greatest feedback K s.t.

$$(H_{\text{spec}} K)^* H_{\text{spec}} \preceq G_{\text{spec}}$$

  for a given overall spec. G_{spec}
- H_{spec} ... low-level spec., i.e., abstraction for plant under low-level control

Result:

- Given overall specification G_{spec}
- Given low-level specifications H_{spec_1}, H_{spec_2}, with $H_{\text{spec}_1} \preceq H_{\text{spec}_2}$ (and some "natural" restrictions in place)
- Compute corresponding optimal feedback control K_{opt_1}, K_{opt_2}
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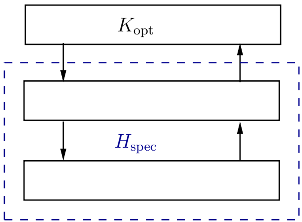

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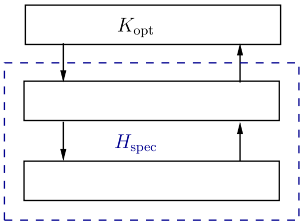

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Conclusions

- Interpreted trade-off between layers in a hierarchical control system from a behavioural point of view
- Formally investigated this trade-off for a specific scenario where top layer is responsible for timing of discrete events
- Resulting setup conveniently described in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$
- Verified that stricter low-level specs indeed allow for more relaxed high-level control

More Details

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More Details (ctd.)

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